Int. J. Heat Mass Transfer. Vol. 24, pp. 251–259
Pergamon Press Ltd. 1981. Printed in Great Britain

0017-9310/81/0201-0251 $02.00/0

VARIABLE TIME STEP METHODS FOR ONE-DIMENSIONAL STEFAN PROBLEM WITH MIXED BOUNDARY CONDITION

R. S. GUPTA and DHIRENDRA KUMAR
Department of Mathematics, University of Roorkee, Roorkee, India

(*Received* 20 *December* 1979 *and in revised form* 9 *July* 1980)

Abstract — The variable time step method introduced by Douglas and Gallie for solving a one-dimensional Stefan problem with constant heat flux at the fixed end is extended to cover a more general boundary condition. The numerical results are obtained for solidification of a liquid initially at its fusion temperature. A method due to Goodling and Khader is discussed in detail and some practical aspects of its implementation are investigated. The same problem is solved by the "modified variable time step" method earlier suggested by the present authors. The results from all the methods are almost identical. An approximate analytical solution is obtained by the heat-balance integral method.

NOMENCLATURE

u,	temperature;
t,	time;
S, s,	positions of interface;
X, x,	space coordinates;
a, b,	arbitrary constants;
A, B,	parameters;
r,	constant parameter.

Greek symbols

α,	thermal diffusivity of solid;
Δt,	time step;
Δx,	space interval;
σ,	parameter.

Subscripts

i, j, p,	locations in x–t plane.

Superscripts

k,	number of iterations.

1. INTRODUCTION

DUE TO their wide range of applications the phase change problems arising during the process of melting/solidification have drawn the considerable attention of mathematicians, engineers and scientists alike in the recent past. These problems are also called "moving boundary problems" since the solid/liquid interface changes its position continuously during the process of phase-change. Sometimes such problems are referred to as "Stefan problems" after Stefan [1] who published his first paper on this subject. Since the boundary condition at the solid/liquid interface in the case of a two phase problem, or at the receding end in the case of a one phase (ablation) problem renders the simple heat conduction equation to a non-linear one, the exact analytical solution is, in general, not possible to obtain. Therefore, recourse is made either to approximate analytical methods or to numerical methods. Amongst the first category the methods due to Goodman [2] and Biot [3] are noteworthy while in the other a number of methods have been suggested from time to time. It will be worthwhile to mention, in particular, the general surveys by Muehlbauer and Sunderland [4] and by Bankhoff [5] on moving boundary problems.

The numerical methods for a one-dimensional Stefan problem have been classified by Gupta [6] as (i) fixed grid methods and (ii) variable grid methods. The fixed grid methods are those in which the space–time domain is subdivided into a finite number of equal grids for all times. At any time the moving boundary lies somewhere between two grid points. The position of the moving boundary and temperatures at its immediate neighbouring points are calculated by using finite difference replacements for unequal intervals. The methods of Crank [7] and Ehrlich [8] may be cited as examples of this class. Under variable grid methods the x–t domain is subdivided into equal intervals in one direction only. The corresponding grid size in the other direction is then determined so that the moving boundary always remains at a grid point. Murray and Landis [9] choose equal steps in time direction and keep the number of space intervals fixed for all times. It should be noted that since the number of space intervals is kept fixed, its size changes (increases or decreases) as the boundary moves. Another example of a variable grid method is that of Douglas and Gallie [10]. They subdivide the x-direction into equal intervals and choose time steps such that the moving boundary crosses exactly one mesh during that interval. Goodling and Khader [11, 12] have also given a variable time step method which has been discussed in greater detail in the present paper. Yet another method is due to Crank and Gupta [13, 14] in which t-direction is subdivided into equal intervals and size of the space intervals is kept fixed. This results in unequal interval near the fixed surface

with other intervals remaining of fixed size throughout the process. There is another important method, called the Isotherm Migration Method (IMM), falling under variable grid methods which calculates the movement of previously selected isotherms into the medium in one time step. Although initially introduced by Chernous'ko [15] the IMM has been systematically presented by Dix and Cizek [16]. This method is later extended in two dimensions by Crank and Gupta [17].

In the present paper we are concerned with variable time step methods only, i.e. when x-direction is subdivided into a finite number of equal intervals and a time step is determined such that the boundary traverses one space mesh during that time. The paper consists of two parts. In Part I the method of Douglas and Gallie [9], originally presented for constant heat flux at the fixed surface, is extended to cover a more general (convective type) boundary condition. We will refer to it as Extension of Douglas and Gallie's (EDG) method. In Part II the "modified variable time step" method [18] proposed earlier by present authors also for a constant heat flux has been applied to solve the same problem. This will be referred to as the MVTS method in future. Numerical results are obtained for both of these methods and the agreement is found to be extremely good. Analytical expressions for movement of the interface and temperature distribution are also obtained using the heat balance integral method of Goodman [2]. The results compare quite well with those from the EDG and MVTS methods.

2. PART I: EXTENSION OF DOUGLAS AND GALLIE'S (EDG) METHOD

The problem

Let us consider the inward solidification of a liquid, initially at its fusion temperature of unity, enclosed by $-1 \le X \le 1$. The liquid is allowed to cool by losing heat through surfaces $X = \pm 1$ according to a convective boundary condition (Fig. 1).

Since the same boundary conditions are applied on both the fixed surfaces $X = \pm 1$, the solidification process will be symmetrical about $X = 0$. We will therefore be concerned with the solution in the region $0 \le X \le 1$ only. The temperature distribution in the other region, i.e. $-1 \le X \le 0$ can be known by symmetry. Mathematically, expressed in its non-dimensional form we require the solution of the equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial X^2}, \quad S(t) < X < 1, \quad t > 0,$$

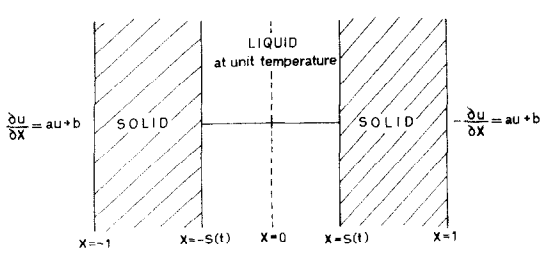

FIG. 1. One-dimensional solidification with convective boundary condition.

along with the boundary conditions

$$-\frac{\partial u}{\partial X} = au + b, \quad X = 1, \quad t > 0;$$

$$u = 1, \quad 0 \le X \le S(t), \quad t \ge 0;$$

$$\frac{dX}{dt} = \frac{\partial u}{\partial X}, \quad X = S(t), \quad t > 0,$$

and the initial condition

$$S(0) = 0,$$

where $S(t)$ denotes the distance of the moving boundary from the fixed surface $X = 1$ at time t.

Transforming the above system by putting $x = 1 - X$ and $s(t) = 1 - S(t)$, the governing equations become

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0; \tag{1}$$

$$\frac{\partial u}{\partial x} = au + b, \quad x = 0, \quad t > 0; \tag{2}$$

$$u = 1, \quad s(t) \le x \le 1, \quad t \ge 0; \tag{3}$$

$$\frac{dx}{dt} = \frac{\partial u}{\partial x}, \quad x = s(t), \quad t > 0; \tag{4}$$

$$s(0) = 0, \tag{5}$$

where s is the distance of the interface from $x = 0$.

A variable time grid

We subdivide the region $0 \le x \le 1$ into, say, n intervals each of width Δx such that $n\Delta x = 1$. The time interval Δt at each step is chosen such that the boundary moves a distance Δx during that interval. Any point (x_i, t_j) in x–t domain is given by

$$\left(i\Delta x, \sum_{m=0}^{j-1} \Delta t_m\right)$$

where Δt_m denotes the time interval in which the boundary moves one space interval from $m\Delta x$ to $(m+1)\Delta x$; t_0 being zero time. Let the moving boundary be at a distance $j\Delta x$ from the fixed surface $x = 0$ at time t_j (Fig. 2). We wish to calculate Δt_j, i.e. the time taken by the moving boundary in traversing a distance

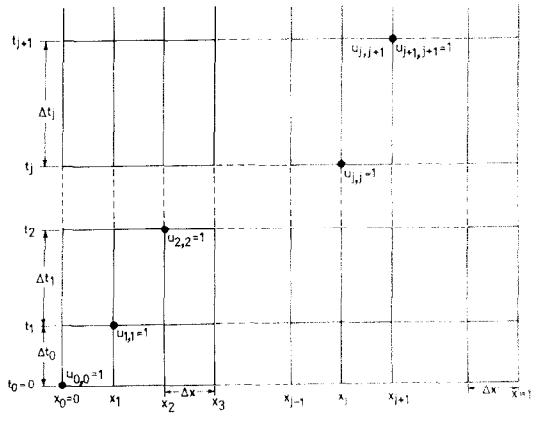

FIG. 2. Variable time grid along with positions of the interface.

Δx from its position $j\Delta x$ to $(j+1)\Delta x$ along with temperatures at the mesh points $x_0, x_1, \ldots, x_{j+1}$ at time t_{j+1}.

Extension of the method and its implementation

Let $u_{i,j}$ represent temperature at the mesh point (i,j) of the variable time grid (Fig. 2). Replacing left side of (1) by backward difference at the point (x_i, t_{j+1}) and right side by central difference, we can write

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t_j} = \alpha \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{(\Delta x)^2}.$$

Rearrangement of above gives

$$-\alpha r u_{i-1,j+1} + (2\alpha r + 1)u_{i,j+1} - \alpha r u_{i+1,j+1} = u_{i,j}, \quad i = 1,2,\ldots,j, \tag{6}$$

where $r = \Delta t_j/(\Delta x)^2$.

In order to determine Δt_j we integrate both sides of equation (1) with respect to x, from 0 to $s(t)$. Making use of (2) through (4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{s(t)} u(x,t)\mathrm{d}x - (\alpha+1)s\right] = -\alpha a u(0,t) - \alpha b.$$

Integrating further with respect to t from 0 to t and using (5) we obtain

$$\alpha b t + \alpha a \int_0^t u(0,t)\mathrm{d}t = (\alpha+1)s - \int_0^{s(t)} u(x,t)\mathrm{d}x. \tag{7}$$

The finite difference replacement of (7), when $t = t_{j+1}$, is made to give

$$\alpha b t_{j+1} + \alpha a \sum_{p=1}^{j+1} u_{0,p}\Delta t_{p-1} = (\alpha+1)(j+1)\Delta x - \Delta x \sum_{i=1}^{j+1} u_{i+1,j+1}. \tag{8}$$

After a little bit of manipulation (8) gives

$$\Delta t_j = \left[(\alpha+1)(j+1)\Delta x - \alpha b t_j - \Delta x \sum_{i=1}^{j+1} u_{i+1,j+1} - \alpha a \sum_{p=1}^{j} u_{0,p}\Delta t_{p-1}\right] \Big/ [\alpha(b + a u_{0,j+1})]. \tag{9}$$

It should be noted that in (8) the finite difference replacements of the integrals have been made such that the value of Δt_j obtained from (9) matches with the one obtained by satisfying the boundary conditions (2) and (4) at $j = 1$ (see Section 4).

By choosing a suitable estimate of Δt_j we compute u's from (6) and the estimated value of Δt_j is subsequently improved by (9). This iterative process is repeated until desired accuracy in Δt_j is achieved. We write the kth iteration for solving $(j+1)$ equations in $(j+1)$ unknowns, viz. $u_{i,j+1}$, $i = 0,1,\ldots,j$ as follows:

$$u^{(k)}_{1,j+1} - (1 + a\Delta x)u^{(k)}_{0,j+1} = b\Delta x \tag{10}$$

$$-\alpha r^{(k)} u^{(k)}_{i-1,j+1} + (2\alpha r^{(k)} + 1)u^{(k)}_{i,j+1} - \alpha r^{(k)} u^{(k)}_{i+1,j+1} = u_{i,j}, \quad i = 1,2,\ldots,j \tag{11}$$

where $r^{(k)} = \Delta t_j^{(k)}/(\Delta x)^2$ and $u^{(k)}_{j+1,j+1} = 1$ from (3) for all k.

Similarly $(k+1)$th iteration for calculating Δt_j can be written from (9) as

$$\Delta t_j^{(k+1)} = \left[(\alpha+1)(j+1)\Delta x - \alpha b t_j - \Delta x \sum_{i=1}^{j+1} u^{(k)}_{i+1,j+1} - \alpha a \sum_{p=1}^{j} u_{0,p}\Delta t_{p-1}\right] \Big/ [\alpha(b + a u^{(k)}_{0,j+1})]. \tag{12}$$

Choosing $\Delta t_j^{(0)}$ same as Δt_{j-1}, already calculated, we determine $u^{(0)}_{i,j+1}$, $i = 0,1,2,\ldots,j$ from (10) and (11) and these values in turn are used to find a new estimate $\Delta t_j^{(1)}$ from (12). $\Delta t_j^{(1)}$ substituted in (10) and (11) gives new set of values of $u^{(1)}$'s. This process is continued until difference between two successive values of Δt_j becomes small to desired accuracy.

3. PART II: MODIFIED VARIABLE TIME STEP (MVTS) METHOD AND ITS COMPARISON WITH OTHER METHODS

Goodling and Khader [11, 12] have given a variable time step method and have applied it to the problem enunciated in Part I. However, numerical values are not tabulated by them and some essential steps for implementing the method are also not explained clearly in any of their papers [11, 12]. Therefore we have computed the results independently and have explained in detail the relevant points for implementation of their method. The same problem is then solved by the MVTS method which was earlier suggested by the present authors [18] for a moving boundary problem with constant heat flux. Let us describe these methods in brief.

(a) *Goodling and Khader's (GK) method*

They arrive at the same set of simultaneous equations (10, 11) for determining the temperatures at $(j+1)$th time level at the kth iteration. The boundary condition (4) is replaced by the following finite difference formula

$$\frac{u^{(k)}_{j+1,j+1} - u^{(k)}_{j,j+1}}{\Delta x} = \frac{\Delta x}{\Delta t_j^{(k)}}$$

giving

$$\Delta t_j^{(k)} = \frac{\Delta x^2}{1 - u^{(k)}_{j,j+1}}. \tag{13}$$

The method suggests that after choosing some value of $u^{(0)}_{j,j+1}$, $\Delta t_j^{(0)}$ is calculated from (13). Taking this value of $\Delta t_j^{(0)}$, $u^{(0)}_{i,j+1}$, $i = 0,1,2,\ldots,j$ are calculated from (11). The boundary condition (10) is then tested for the accuracy for the selection of $u^{(0)}_{j,j+1}$. If the boundary condition (10) is not satisfied, $u_{j,j+1}$ is estimated again and a new Δt_j is obtained from (13) which in turn is used to obtain new values of $u_{i,j+1}$. This process is repeated until a desired accuracy in (10) is achieved.

The manner in which $u_{j,j+1}$ is to be selected has not been elaborated by them. However, we tried taking

$u^{(0)}_{j,j+1}$ equal to $u_{j-1,j}$ as a suitable estimate; kept on giving a small increment to it successively until a change of sign in the error in (10) is noticed; and interpolation for $u_{j,j+1}$ is then carried out either by the method of chords or the bisection method. We find that this procedure does not work in the present problem after some time. The method for solving (11) becomes unstable and the error in boundary condition (10) behaves in an irregular fashion. Therefore we discard this method of selecting $u_{j,j+1}$ and adopt the following approach.

Instead of selecting $u_{j,j+1}$, we estimate $\Delta t^{(0)}_j$ from

$$\Delta t^{(0)}_j = \Delta t_{j-1} + (\Delta t_{j-1} - \Delta t_{j-2}) + \varepsilon \qquad (14)$$

where ε is small. $u^{(0)}_{j,j+1}$ is calculated using (13); values of $u^{(0)}_{i,j+1}$ are then determined from (11); boundary condition (10) is checked whether it is satisfied within the desired accuracy; if not, a suitable decrement, approximately 0.2% of Δx is given to Δt_j and calculations repeated until there is a change of sign in the error; the value of Δt_j is then interpolated by the method of bisection.

(b) *Modified variable time step (MVTS) method*

In this method also we make finite difference replacements of (1) at (x_i, t_{j+1}) and obtain (11) as in EDG and GK methods. The boundary condition (10) is also written as before. The interface condition (4) is written, similar to (13), as

$$\Delta t^{(k+1)}_j = \frac{\Delta x^2}{1 - u^{(k)}_{j,j+1}}. \qquad (15)$$

Choosing $\Delta t^{(0)}_j$ equal to Δt_{j-1} initially, we solve the set of simultaneous equations (10, 11) which gives $u^0_{i,j+1}$, $i = 0, 1, 2, \ldots, j$. Using value of $u^{(0)}_{j,j+1}$ in (15) we get first estimate $\Delta t^{(1)}_j$. This process is repeated until desired accuracy in Δt_j is obtained. It should be noted that in the present method, accuracy check is made at the interface while in the GK method it is done at the fixed end. This results in solving the simultaneous equations by Gaussian elimination in the present method whereas an iterative technique, which makes the process unstable, is used in the GK method.

4. NUMERICAL COMPUTATIONS

To start any of the methods from $t = 0$ to $t = \Delta t$, i.e. to calculate Δt_0, we make the following finite difference replacements of the boundary conditions at the fixed surface and the moving boundary at $t = \Delta t$, giving,

$$\frac{u_{1,1} - u_{0,1}}{\Delta x} = a u_{0,1} + b,$$

and

$$\frac{u_{0,1} - u_{1,1}}{\Delta x} = \frac{\Delta x}{\Delta t_0}$$

respectively. Eliminating $u_{0,1}$ and remembering that $u_{1,1} = 1$ we get from the above

$$\Delta t_0 = \frac{\Delta x(1 + a\Delta x)}{(a + b)}. \qquad (16)$$

In order to compare our results we take a sample problem with $\alpha = 1$ in (1) and $a = 10$, $b = 0$ in (2), the case which has been dealt with by Goodling and Khader [11, 12]. Values of Δt and temperature distributions have been computed from all the three methods, viz. EDG, GK and MVTS for $\Delta x = 0.100$, 0.050, 0.025 and 0.010. A maximum error of 0.05% in Δt is allowed in the EDG and MVTS methods while in the GK method the same error is allowed in the boundary condition (10). It has been noticed that for $\Delta x = 0.01$, the GK method breaks down when boundary condition is allowed to satisfy within an accuracy of 0.05% or 0.10%, therefore the results in it for $\Delta x = 0.01$ are computed with a larger error (0.5%). In all the methods Δt_0 has been calculated from (16).

Table 1 gives comparative figures, for all the methods, for the time required by the interface to move one space interval as well as surface temperatures. As all these methods are iterative, the number of iterations for obtaining a Δt are shown in parentheses in the Δt-column. For example, when $s(t) = 0.20$ and $\Delta x = 0.05$, the time taken by the interface in moving from $s(t) = 0.15$ to 0.20 is 0.0186 by EDG, 0.0187 by MVTS and 0.0186 by the GK method and the corresponding number of iterations are 3, 3 and 8 respectively. Table 2 shows temperature distributions in the solid region at different times for $\Delta x = 0.05$ from various methods.

5. DISCUSSION

It can be seen from Tables 1 and 2 that the EDG method has worked very well throughout. From Table 1 we see that the method converges quite fast. The value of Δt, within prescribed accuracy, is obtained in 3 iterations or less for most part of the solidification process. At the last stage only, the number of iterations sometimes go up to a maximum of 6. The numerical results as well as the number of iterations to obtain Δt from the MVTS method compare very well with the EDG method. Although the results obtained from the GK method agree quite well with the other two methods, the iterations are much larger.

Further, it may be recollected that numerical results from the GK method corresponding to $\Delta x = 0.01$ have been computed by allowing a relatively larger error. It must be emphasized that the method of estimating $u_{j,j+1}$ and its subsequent adjustment in the GK method is very crucial. The method may fail (not converge) or may give inaccurate results if a different initial estimate is chosen. We have further refined the method by selecting an estimate for Δt rather than for $u_{j,j+1}$ [see equation (14)].

Figure 3 shows that the movement of the interface is somewhat faster in the beginning in comparison to later part of the process when it assumes an almost linear relationship with time. It also indicates that solidification process is completed at $t \approx 0.80$ when the

Table 1. Comparison of time (Δt) required by the interface for moving a space interval (Δx) along with surface temperature $u(0,t)$

Position of interface $s(t)$	$\Delta x = 0.100$		$\Delta x = 0.050$		$\Delta x = 0.025$		$\Delta x = 0.010$	
	Δt	$u(0,t)$	Δt	$u(0,t)$	Δt	$u(0,t)$	Δt	$u(0,t)$
	0.0200 (0)	0.5000	0.0116 (2)	0.5232	0.0061 (2)	0.5280	0.0025 (2)	0.5299
0.1	0.0200 (0)	0.5000	0.0116 (3)	0.5231	0.0061 (3)	0.5280	0.0025 (2)	0.5299
	0.0200 (0)	0.5000	0.0116 (7)	0.5232	0.0061 (7)	0.5280	0.0025 (5)	0.5299
	0.0356 (3)	0.3596	0.0186 (3)	0.3635	0.0095 (3)	0.3641	0.0038 (2)	0.3642
0.2	0.0356 (4)	0.3596	0.0187 (3)	0.3634	0.0095 (3)	0.3641	0.0038 (2)	0.3642
	0.0356 (9)	0.3596	0.0186 (8)	0.3635	0.0095 (8)	0.3641	0.0038 (6)	0.3642
	0.0494 (3)	0.2770	0.0253 (3)	0.2774	0.0128 (3)	0.2771	0.0052 (3)	0.2769
0.3	0.0494 (4)	0.2770	0.0254 (3)	0.2773	0.0128 (3)	0.2771	0.0052 (2)	0.2769
	0.0494 (14)	0.2770	0.0253 (7)	0.2773	0.0128 (3)	0.2771	0.0052 (2)	0.2771
	0.0627 (3)	0.2242	0.0319 (3)	0.2237	0.0161 (3)	0.2233	0.0065 (3)	0.2231
0.4	0.0627 (4)	0.2242	0.0319 (3)	0.2237	0.0161 (3)	0.2233	0.0065 (2)	0.2231
	0.0627 (9)	0.2242	0.0319 (3)	0.2237	0.0161 (6)	0.2234	0.0065 (2)	0.2233
	0.0759 (3)	0.1879	0.0385 (3)	0.1873	0.0194 (3)	0.1869	0.0078 (3)	0.1867
0.5	0.0759 (4)	0.1879	0.0385 (3)	0.1873	0.0194 (3)	0.1869	0.0078 (2)	0.1867
	0.0759 (6)	0.1879	0.0385 (7)	0.1873	0.0194 (8)	0.1869	0.0078 (8)	0.1866
	0.0891 (3)	0.1616	0.0450 (3)	0.1609	0.0226 (3)	0.1606	0.0091 (6)	0.1605
0.6	0.0890 (4)	0.1616	0.0450 (3)	0.1609	0.0226 (3)	0.1606	0.0091 (2)	0.1604
	0.0891 (5)	0.1616	0.0450 (4)	0.1610	0.0226 (7)	0.1606	0.0091 (9)	0.1602
	0.1021 (3)	0.1416	0.0515 (3)	0.1410	0.0259 (3)	0.1407	0.0104 (2)	0.1407
0.7	0.1021 (4)	0.1416	0.0516 (3)	0.1410	0.0259 (3)	0.1407	0.0104 (2)	0.1406
	0.1021 (3)	0.1417	0.0515 (4)	0.1411	0.0259 (7)	0.1407	0.0104 (8)	0.1406
	0.1152 (3)	0.1260	0.0580 (3)	0.1255	0.0291 (3)	0.1252	0.0117 (2)	0.1252
0.8	0.1152 (4)	0.1260	0.0581 (3)	0.1255	0.0291 (2)	0.1252	0.0117 (3)	0.1251
	0.1152 (5)	0.1260	0.0581 (7)	0.1255	0.0291 (9)	0.1252	0.0117 (9)	0.1251
	0.1282 (3)	0.1135	0.0646 (3)	0.1130	0.0324 (3)	0.1128	0.0130 (5)	0.1128
0.9	0.1282 (4)	0.1135	0.0646 (3)	0.1130	0.0324 (2)	0.1128	0.0130 (3)	0.1127
	0.1282 (4)	0.1134	0.0646 (7)	0.1130	0.0324 (10)	0.1128	0.0130 (12)	0.1126
	0.1413 (3)	0.1032	0.0711 (3)	0.1028	0.0356 (3)	0.1026	0.0142 (6)	0.1026
1.0	0.1413 (4)	0.1032	0.0711 (3)	0.1028	0.0357 (2)	0.1026	0.0143 (1)	0.1026
	0.1413 (6)	0.1032	0.0711 (8)	0.1028	0.0356 (9)	0.1026	0.0143 (14)	0.1026

The upper entry corresponds to EDG method, the middle to MVTS method and the lower to GK method.

Table 2. Comparison of temperatures and times required for the movement of interface

Interface $s(t)$ \ x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	Time t
	0.5232	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0191
0.1	0.5231	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0191
	0.5232	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0191
	0.3635	0.7137	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0529
0.2	0.3634	0.7137	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0530
	0.3635	0.7137	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0530
	0.2774	0.5492	0.7929	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1003
0.3	0.2773	0.5491	0.7928	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1004
	0.2773	0.5492	0.7929	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1003
	0.2237	0.4447	0.6510	0.8373	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1608
0.4	0.2237	0.4446	0.6510	0.8372	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1610
	0.2237	0.4447	0.6510	0.8372	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.1608
	0.1873	0.3730	0.5501	0.7153	0.8658	1.0	1.0	1.0	1.0	1.0	1.0	0.2345
0.5	0.1873	0.3729	0.5501	0.7153	0.8658	1.0	1.0	1.0	1.0	1.0	1.0	0.2347
	0.1873	0.3730	0.5501	0.7153	0.8658	1.0	1.0	1.0	1.0	1.0	1.0	0.2345
	0.1609	0.3209	0.4755	0.6224	0.7597	0.8858	1.0	1.0	1.0	1.0	1.0	0.3213
0.6	0.1609	0.3209	0.4755	0.6224	0.7596	0.8858	1.0	1.0	1.0	1.0	1.0	0.3215
	0.1610	0.3209	0.4755	0.6224	0.7596	0.8858	1.0	1.0	1.0	1.0	1.0	0.3213
	0.1410	0.2814	0.4182	0.5498	0.6748	0.7920	0.9006	1.0	1.0	1.0	1.0	0.4211
0.7	0.1410	0.2814	0.4182	0.5498	0.6748	0.7920	0.9006	1.0	1.0	1.0	1.0	0.4213
	0.1411	0.2814	0.4182	0.5498	0.6748	0.7920	0.9006	1.0	1.0	1.0	1.0	0.4211
	0.1255	0.2505	0.3731	0.4919	0.6061	0.7146	0.8168	0.9120	1.0	1.0	1.0	0.5339
0.8	0.1255	0.2505	0.3730	0.4919	0.6060	0.7146	0.8167	0.9120	1.0	1.0	1.0	0.5342
	0.1255	0.2505	0.3730	0.4919	0.6060	0.7146	0.8167	0.9120	1.0	1.0	1.0	0.5339
	0.1130	0.2257	0.3366	0.4447	0.5495	0.6500	0.7457	0.8362	0.9211	1.0	1.0	0.6598
0.9	0.1130	0.2257	0.3366	0.4447	0.5494	0.6500	0.7457	0.8362	0.9211	1.0	1.0	0.6601
	0.1130	0.2257	0.3365	0.4447	0.5494	0.6500	0.7457	0.8362	0.9211	1.0	1.0	0.6598
	0.1028	0.2053	0.3065	0.4057	0.5022	0.5955	0.6852	0.7708	0.8520	0.9284	1.0	0.7987
1.0	0.1028	0.2053	0.3065	0.4056	0.5022	0.5955	0.6852	0.7708	0.8520	0.9284	1.0	0.7990
	0.1028	0.2053	0.3065	0.4056	0.5022	0.5955	0.6852	0.7708	0.8519	0.9284	1.0	0.7987

For each value of $s(t)$ the upper entry corresponds to EDG, the middle to MVTS and the lower to GK method ($\Delta x = 0.05$ for each).

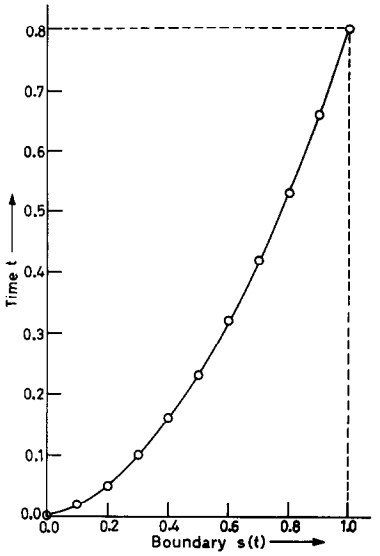

FIG. 3. Position of interface (moving boundary) vs time graph (from EDG with $\Delta x = 0.05$).

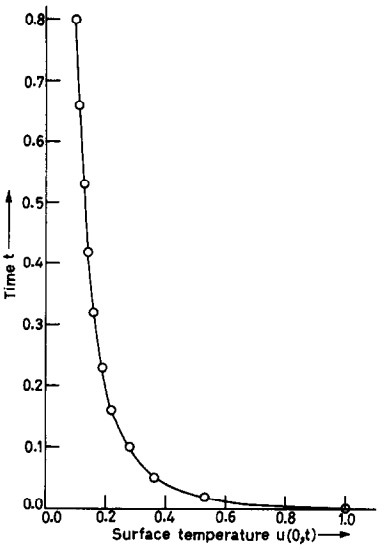

FIG. 5. Plot of surface temperature against time (from EDG with $\Delta x = 0.05$).

interface $s(t)$ has reached $x = 1.0$. In Fig. 4, curves for temperature distributions in the solid region are drawn for various times. A graph between surface temperature and time is plotted in Fig. 5 which shows an appreciable drop in the beginning of the solidification process. Very soon, approximately after $t = 0.1$ the rate of fall of surface temperature gets exceedingly small.

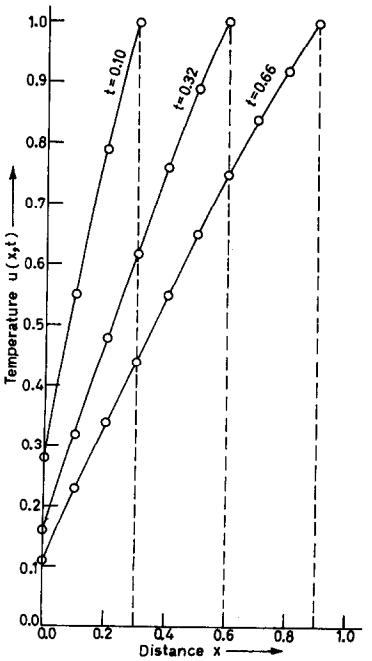

FIG. 4. Temperature distributions at various times (from EDG with $\Delta x = 0.05$). Dotted lines show positions of the interface.

Figures 3–5 have been drawn from the numerical results of the EDG method with $\Delta x = 0.05$ (Table 2). It may be observed that these values are almost identical to those obtained from the other methods with corresponding Δx.

The variable time step methods need special attention when the movement of the boundary is very slow. However, in the present case since the boundary is moving quite fast throughout the process of solidification (Fig. 3), the time calculated for a particular position of the interface does not differ very much even when we have taken different values of Δx (Table 1).

6. INTEGRAL METHOD

Using Goodman's integral method [2] we have obtained an approximate analytical expression for the temperature distribution

$$u(x,t) = 1 + A(x - s) + B(x - s)^2 \qquad (17)$$

where

$$A = \frac{-\sigma + (3\sigma^2 - 2)^{1/2}}{s(\sigma + 1)}$$

$$B = -\frac{1}{s^2(\sigma + 1)^2}[2\sigma^2 - \sigma(3\sigma^2 - 2)^{1/2} - 1]$$

and

$$\sigma = 1 + 10s.$$

The relationship between position of the interface and time is given by the differential equation,

$$\frac{dt}{d\sigma} = \frac{(30^2-2)^{1/2}\{7\sigma^4+28\sigma^3+30\sigma^2+12\sigma+3\}+\{3\sigma^5+12\sigma^4+30\sigma^3+12\sigma^2-25\sigma-16\}}{600(\sigma+1)^2(3\sigma^2-2)^{1/2}\{3\sigma+2-(3\sigma^2-2)^{1/2}\}} \tag{18}$$

The analytical solution of (18) does not seem feasible, therefore, we solve it numerically by using a fourth order Runge–Kutta algorithm. Table 3 gives position of the interface $s(t)$ and the surface temperature $u(0,t)$ at various times. Corresponding numerical values are also given from EDG and MVTS methods for $\Delta x = 0.05$ for comparison.

Table 3. Comparison of time t required for a movement $s(t)$ of the interface and the surface temperature $u(0,t)$ from the Integral Method with EDG and MVTS methods

Position of interface $s(t)$	Time t			Surface temperature $u(0,t)$		
	EDG method	MVTS method	Integral method	EDG method	MVTS method	Integral method
0.1	0.0191	0.0191	0.0178	0.5232	0.5231	0.5375
0.2	0.0529	0.0530	0.0487	0.3635	0.3634	0.3750
0.3	0.1003	0.1004	0.0918	0.2774	0.2773	0.2887
0.4	0.1608	0.1610	0.1468	0.2237	0.2237	0.2349
0.5	0.2345	0.2347	0.2136	0.1873	0.1873	0.1980
0.6	0.3213	0.3215	0.2920	0.1609	0.1609	0.1712
0.7	0.4211	0.4213	0.3820	0.1410	0.1410	0.1508
0.8	0.5339	0.5342	0.4836	0.1255	0.1255	0.1348
0.9	0.6598	0.6601	0.5967	0.1130	0.1130	0.1218
1.0	0.7987	0.7990	0.7214	0.1028	0.1028	0.1111

Note—Values for EDG and MVTS methods correspond to $\Delta x = 0.05$.

Acknowledgement — The financial assistance received by one of the authors (Dhirendra Kumar) from the University Grants Commission under the F.I.P. Scheme is acknowledged with thanks.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd edn. Clarendon Press, Oxford (1959).
2. T. R. Goodman, The heat-balance integral and its applications to problems involving change of phase, *Trans. Am. Soc. Mech. Engrs* **80,** 335–342 (1958).
3. M. A. Biot, Further developments of new methods in heat flow analysis, *J. Aerospace Sci.* **26,** 367–381 (1959).
4. J. C. Muehlbauer and J. E. Sunderland, Heat conduction with freezing or melting, *Appl. Mech. Rev.* **18,** 951–959 (1965).
5. S. G. Bankhoff, Heat conduction or diffusion with change of phase, in *Advances in Chemical Engineering*, pp. 75–150 (edited by T. B. Drew, J. W. Hoops and T. Vermeulen). Academic Press, New York (1964).
6. R. S. Gupta, Some numerical and integral methods for solving moving boundary problems in heat flow and diffusion, Ph.D. Thesis, Brunel University, Uxbridge (1972).
7. J. Crank, Two methods for the numerical solution of moving boundary problems in diffusion and heat flow, *J. Mech. Appl. Math.* **10,** 220–231 (1957).
8. L. W. Ehrlich, A numerical method of solving a heat flow problem with moving boundary, *J. Assn Comp. Math.* **5,** 161–176 (1958).
9. W. D. Murray and F. Landis, Numerical and machine solutions of transient heat conduction problems involving melting or freezing, *J. Heat Transfer* **81,** 106–112 (1959).
10. J. Douglas and T. M. Gallie, On the numerical integration of a parabolic differential equation subject to a moving boundary condition, *Duke Math. Jl* **22,** 557–570 (1955).
11. J. S. Goodling and M. S. Khader, One dimensional inward solidification with a convective boundary condition, *AFS Cast Metals Res. Jl* **10,** 26–29 (1974).
12. J. S. Goodling and M. S. Khader, Inward solidification with radiation-convection boundary condition, *J. Heat Transfer* **96,** 114–115 (1974).
13. John Crank and R. S. Gupta, A method for solving moving boundary problems in heat flow using cubic splines or polynomials, *J. Inst. Maths Applics* **10,** 296–304 (1972).
14. Radhey S. Gupta, Moving grid method without interpolations, *Comp. Meth. Appl. Mech. Engng* **4,** 143–152 (1974).
15. F. L. Chernous'ko, Solution of non-linear heat conduction problems in media with phase changes, *Int. Chem. Engng* **10,** 42–48 (1970). [First published in *Zh. Prikl. Mekh. Tekh. Fiz.* No. 2, 6–14 (1969)].
16. R. C. Dix and J. Cizek, The isotherm migration method for transient heat conduction analysis, *Proc. Fourth Int. Heat Transf. Conf., Paris*, Vol. 1. A.S.M.E., New York (1971).
17. J. Crank and Radhey S. Gupta, Isotherm migration method in two dimensions, *Int. J. Heat Mass Transfer* **18,** 1101–1107 (1975).
18. R. S. Gupta and D. Kumar, A modified variable time step method for one-dimensional Stefan problem, *Comp. Meth. Appl. Mech. Engng* **23,** 101–109 (1980).

METHODES A PAS DE TEMPS VARIABLE POUR LE PROBLEME DE STEFAN A UNE DIMENSION, AVEC DES CONDITIONS AUX LIMITES MIXTES

Résumé—La méthode à pas de temps variable introduite par Douglas et Callie pour résoudre un problème monodimensionnel de Stefan avec un flux thermique constant à l'extrémité fixe est étendue pour couvrir une condition limite plus générale. Les résultats numériques sont obtenus pour la solidification d'un liquide initialement à sa température de fusion. Une méthode due à Goodling et Klader est discutée en détail et quelques aspects pratiques d'application sont étudiés. Le même problème est résolu par la "méthode à pas de temps variable modifiée" antérieurement suggérée par les auteurs. Les résultats de toutes ces méthodes sont à peu près identiques. Une solution analytique approchée est obtenue par la méthode du bilan intégral de chaleur.

METHODEN MIT VARIABLEM ZEITSCHRITT ZUR LÖSUNG DES EINDIMENSIONALEN STEFAN-PROBLEMS MIT GEMISCHTEN RANDBEDINGUNGEN

Zusammenfassung — Die von Douglas und Gallie eingeführte Zeitschritt-Methode zur Lösung des eindimensionalen Stephan-Problems mit konstantem Wärmestrom am festen Ende wird so erweitert, daß sie für eine allgemeine Randbedingung anwendbar wird. Numerische Resultate wurden für das Erstarren einer anfänglich auf Schmelztemperatur befindlichen Flüssigkeit ermittelt. Die Methode von Goodling und Khader wird ausführlich diskutiert, und einige Gesichtspunkte ihrer praktischen Durchführung werden erörtert. Dasselbe Problem wird mit der von den Autoren schon früher vorgeschlagenen variablen Zeitschritt-Methode gelöst. Die Ergebnisse aller Verfahren sind fast identisch. Eine analytische Näherungslösung wurde mit der Wärmebilanzintegral-Methode erhalten.

МЕТОДЫ ПЕРЕМЕННЫХ ВРЕМЕННЫХ ШАГОВ В ЗАДАЧЕ СТЕФАНА СО СМЕШАННЫМИ ГРАНИЧНЫМИ УСЛОВИЯМИ

Аннотация — Метод переменных временных шагов, предложенный Дугласом и Галли для решения одномерной задачи Стефана с постоянным тепловым потоком на фиксированной границе, обобщен на случай более общего граничного условия. Получены численные результаты по затвердеванию жидкости, первоначально находившейся при температуре плавления. Дан подробный анализ метода Гудлинга и Кхадера и рассмотрены некоторые практические аспекты его применения. Та же задача решена методом «модифицированного переменного временного шага», который был ранее предложен авторами настоящей работы. Результаты, полученные всеми перечисленными методами, оказались почти идентичными. Приближенное аналитическое решение получено интегральным методом теплового баланса.